FALL 2024: MATH 790 EXAM 2 SOLUTIONS

Throughout this exam, V will denote a finite dimensional vector space over the field F. Each problem is worth 10 points. You may use the Daily Update, your notes from class or any homework you have done, **but you may not use any other resources**, including your book, any other book, any information taken from the internet, nor may you consult with any students or professors, other than your Math 790 professor. You may freely use basic linear algebra facts presented in a first course on linear algebra, but you **may not** use any advanced linear algebra facts not presented in class. Please turn your solutions in at the start of class on Monday, November 11. **Good luck on the exam!**

1. Let $T \in \mathcal{L}(V, V)$ and suppose there exists $v \in V$ satisfying $V = \langle T, v\rangle$.
   - (i) Let $U \subseteq V$ be a T-invariant subspace. Prove that U is cyclic with respect to T, i.e., $U = \langle T, u\rangle$, for some $u \in U$. Hint: Consider $u := a(T)(v)$, where $a(x)$ is the monic polynomial of least degree such that $a(T)(v) \in U$.
   - (ii) Suppose $\mu_T(x) = p_1(x) \cdots p_r(x)$, where each $p_i(x)$ is irreducible over F and $\deg(\mu_T(x)) = \dim(V)$. Prove that V has exactly 2^r invariant subspaces. Hint: Study the roof of the primary decomposition theorem for T.

Solution. (i) Among all polynomial $f(x) \in F[x]$ with $f(T)(v) \in U$, chose $a(x)$ monic of least degree with this property. Note, such a polynomial exists, because the set of monic polynomials $f(x)$ with $f(T)(v) \in U$ is not empty. (Can you see why?) Set $u := a(T)(v)$, and take $u_0 \in U$. Then since $V = \langle T, v\rangle$, $u_0 = g(T)(v)$, for some $g(x) \in F[x]$. Using the division algorithm, we may write $g(x) = a(x)b(x) + r(x)$, where $r(x) = 0$ or $r(x)$ has degree strictly less than the degree of $a(x)$. Suppose $r(x)$ is not zero. Then

$$u_0 = g(T)(v) = a(T)b(T)(v) + r(T)(v) = b(T)(u) + r(T)(v).$$

Thus, $r(T)(v) = u_0 - b(T)(u)$. Since U is T-invariant, $b(T)(u) \in U$. Thus, $r(T)(v) \in U$, and this contradicts the minimality of $a(x)$. Thus, $r(x) = 0$, and $g(x) = b(x)a(x)$. Thus, $u_0 = g(T)(v) = b(T)(u) \in \langle T, u\rangle$, which shows that U is T-cyclic generated by u.

For part (ii), We have $V = W_1 \bigoplus \cdots \bigoplus W_r$, where $W_i = \ker(p_i(T))$. It suffices to show that if U is a T-invariant subspace of V, then $U = W_{i_1} \bigoplus \cdots \bigoplus W_{i_t}$, for some $1 \leq i_1 < \cdots < i_t \leq r$, for counting $\{0\}$ and V, there are 2^r such subspaces.

We first note that, by part (i), each W_i is T-cyclic and any T-invariant subspace $U \subseteq V$ is T-cyclic. Fix a T-invariant subspace U. Suppose $W_i = \langle T, w_i\rangle$ and $U = \langle T, u\rangle$. We have $\mu_{T,u}(x)$ divides $\mu_T(x)$. From the definition of $\mu_T(x)$, this gives $\mu_{T,u}(x) = p_{i_1}(x) \cdots p_{i_t}(x)$, for some $1 \leq i_1 < \cdots < i_t \leq r$.

As in the proof of the primary decomposition theorem, we set $q_j(x) := \Pi_{i_s \neq i_j} p_{i_s}(x)$, for $1 \leq j \leq t$. Then $q_1(x), \dots, q_t(x)$ have no common divisor, thus, there exist $c_1(x), \dots, c_t(x) \in F[x]$ such that

$$1 = c_1(x)q_1(x) + \cdots + c_t(x)q_t(x).$$

Thus, $u = c_1(T)q_1(T)(u) + \cdots + c_t(T)q_t(T)(u)$. By definition, $p_{ij}(T)q_j(T)(u) = 0$, thus, each $c_j(T)q_j(T)(u) \in W_{i_j}$. It follows that $U \subseteq W_{i_1} + \cdots + W_{i_t}$. On the other hand,

$$\begin{aligned}
\dim(U) &= \deg(\mu_{T,u}(x)) \\
&= \deg(p_{i_1}(x)) + \cdots + \deg(p_{i_t}(x)) \\
&= \dim(W_{i_1}) + \cdots + \dim(W_{i_t}) \\
&= \dim(W_{i_1} \bigoplus \cdots \bigoplus W_{i_t}),
\end{aligned}$$

which gives $U = W_{i_1} \bigoplus \cdots \bigoplus W_{i_t}$, as required. $\square$

2. Let A be an 6×6 matrix with entries in $\mathbb{R}$. Prove that there do not exist invertible 6×6 invertible matrices P, Q such that

$$P^{-1}\begin{pmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{pmatrix} P = A = Q^{-1}\begin{pmatrix} C_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_3 \end{pmatrix} Q,$$

where $B_1 = C((x^2+1)^2), B_2 = C(x^2+1)$ and $C(x^2+1) = C_1 = C_2 = C_3$.

Solution. If we set $B := \begin{pmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{pmatrix}$ and $C := \begin{pmatrix} C_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_3 \end{pmatrix}$, then A and B have the same minimal polynomial, and A and$C$ have the same minimal polynomial. Thus, B and C have the same minimal polynomial. But $\mu_B(x) = (x^2+1)^2$ and $\mu_C(x) = x^2+1$. Thus, we cannot have $P^{-1}BP = Q^{-1}CQ$. □

3. Let V and W be finite dimensional inner product spaces over $\mathbb{C}$. For $T \in \mathcal{L}(V, W)$, define $T^* \in \mathcal{L}(W, V)$ as follows. Fix orthonormal bases $B \subseteq V$ and $C \subseteq W$. Then T^* is defined by the equation $[T^*]_C^B = A^*$, where $A = [T]_B^C$. Prove that:

(i) T^* is well-defined, i.e., we get the same linear transformation if we take orthonormal bases $B' \subseteq V$ and $C' \subseteq W$.
(ii) $\langle T(v), w\rangle_W = \langle v, T^*(w)\rangle_V$, for all $v \in V$ and $w \in W$.
iii) $(\lambda T)^* = \overline{\lambda} T^*$, for all $\lambda \in \mathbb{C}$.

Solution. For (i), let $T' : W \to V$ be edfined by the equation $[T']_{C'}^{B'} = ([T]_{B'}^{C'})^*$. Then we have

$$\begin{aligned}
[T']_C^B &= [I]_{B'}^B \cdot [T']_{C'}^{B'} \cdot [I]_C^{C'} \\
&= [I]_{B'}^B \cdot ([T]_{B'}^{C'})^* \cdot [I]_C^{C'} \\
&= ([I]_B^{B'})^* \cdot ([T]_{B'}^{C'})^* \cdot ([I]_C^{C'})^* \\
&= ([I]_{C'}^{C} \cdot [T]_{B'}^{C'} \cdot [I]_B^{B'})^* \\
&= ([T]_B^C)^* \\
&= [T^*]_B^C,
\end{aligned}$$

which shows $T' = T^*$.

For (ii): Set $B := \{b_1, \ldots, b_n\}$ and $C := \{c_1, \ldots, c_m\}$ and write $A = (a_{ij})$, for $[T]_B^C$. We'll drop V and W from the inner product notation, with the understanding that there are two different inner products, one for V and one for W. Then:

$$\langle T(b_i), c_j\rangle = \langle a_{1i}c_1 + \cdots + a_{mi}c_m, c_j\rangle = a_{ji} = \langle b_i, \overline{a_{j1}}b_1 + \cdots + \overline{a_{jn}}b_n\rangle = \langle b_i, T^*(c_j)\rangle,$$

since B and C are orthonormal bases. Now suppose $v = \Sigma_i \alpha_i b_i$ and $w = \Sigma_j \beta_j c_j$. Then,

$$\langle T(v), w\rangle = \Sigma_{i,j}\alpha_i\overline{\beta_j}\langle T(b_i), c_j\rangle = \Sigma_{i,j}\alpha_i\overline{\beta_j}\langle b_i, T^*(c_j)\rangle = \langle v, T^*(w)\rangle$$

For (iii): $[\lambda T]_B^C = \lambda A$, so that $[(\lambda T)^*]_C^B = (\lambda A)^* = \overline{\lambda}A^* = \overline{\lambda}[T^*]_C^B$, which gives what we want. □

4. Maintaining the notation as in the previous problem, prove:

(i) T^*T is self-adjoint.
(ii) $\ker(T^*T) = \ker(T)$.
(iii) $\mathrm{im}(T^*T) = \mathrm{im}(T^*)$.
(iv) $\mathrm{rank}(T) = \mathrm{rank}(T^*) = \mathrm{rank}(T^*T)$.

Solution. For (i), we first note $[T^*T]_B^B = [T^*]_C^B \cdot [T]_B^C = A^*A$. Thus,

$$[(T^*T)^*]_B^B = ([T^*T]_B^B)^* = (A^*A)^* = A^*A = [T^*T]_B^B,$$

so $T^*T = (T^*T)^*$.

For (ii): Clearly $\ker(T) \subseteq \ker(T^*T)$. Conversely, if $v \in \ker(T^*T)$, then $0 = \langle v, T^*T(v)\rangle = \langle T(v), T(v)\rangle$, so $T(v) = 0$, showing $\ker(T^*T) \subseteq \ker(T)$, which gives what we want.

For (iii): We begin with an observation: $w \in \ker(T^*)$ if and only if $T^*(w) = 0$ if and only if $\langle v, T^*(w)\rangle = 0$, for all $v \in V$ if and only if $\langle T(v), w\rangle = 0$, for all $v \in V$ if and only if $w \in \operatorname{im}(T)^\perp$. Thus, $\ker(T^*) = \operatorname{im}(T)^\perp$, so that $\ker(T^*)^\perp = \operatorname{im}(T)$. Thus, $W = \ker(T^*) \bigoplus \operatorname{im}(T)$. Now, clearly, $\operatorname{im}(T^*T) \subseteq \operatorname{im}(T^*)$. Suppose $w \in W$, so $T^*(w) \in \operatorname{im}(T^*)$. We can write $w = u + T(v)$, where $u \in \ker(T^*)$. Thus, $T^*(w) = T^*(u) + T^*T(v) = T^*T(v)$, so that $T^*(w) \in \operatorname{im}(T^*T)$, which gives what we want. □

For (iv): That $\operatorname{rank}(T^*) = \operatorname{rank}(T^*T)$ follows immediately from (iii). On the other hand,

$$\begin{aligned} \operatorname{rank}(T^*T) &= \dim(V) - \operatorname{nullity}(T^*T) \\ &= \dim(V) - \operatorname{nullity}(T) \quad (\text{by } (ii)) \\ &= \operatorname{rank}(T). \end{aligned}$$

5. Give an example of a matrix $A \in \mathrm{M}_2(\mathbb{C})$ that is not self-adjoint, but A is normal, and its entries are not in $\mathbb{R}$. Then show that, for you particular choice of A, $||Av|| = ||A^*v||$, for all $v \in \mathbb{C}^2$.

Solution. Set $A := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, so that $A^* = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$. Then $A^*A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = AA^*$, so that A is a normal matrix. Let $v \in \mathbb{C}^2$, $v = \begin{pmatrix} a \\ b \end{pmatrix}$.Then $Av = \begin{pmatrix} ib \\ ia \end{pmatrix}$ and $A^*v = \begin{pmatrix} -ib \\ -ia \end{pmatrix}$. Thus, $||Av|| = b^2 + a^2 = ||A^*v||$.

6. For the matrix $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$, find invertible 4×4 matrices over $\mathbb{R}$ such that $P^{-1}AP = R_1$ and $Q^{-1}AQ = R_2$, where R_1 is the invariant factor rational canonical form of A and R_2 is the elementary divisor rational canonical form of A.

Solution It should have been noted that $A \in \mathrm{M}_4(\mathbb{R})$. Then, $\chi_A(x) = (x-2)(x+1)(x^2+1) = x^4 - x^3 - x^2 - x - 2$ has three irreducible factors. Since $\chi_A(x)$ and $\mu_A(x)$ have the same irreducible factors, $\chi_A(x) = \mu_A(x)$. Thus, the invariant factor form of the RCF of A is $R_1 = C(\mu_A(x)) = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. To find P such that $P^{-1}AP = R_1$, we need to find a maximal vector $v \in \mathbb{R}^4$ for A, and then P will be the matrix whose columns are v, Av, A^2v, A^3v. To find v, we find the maximal vectors for the primary component decomposition. In other words, we have $\mathbb{R}^4 = W_1 \bigoplus W_2 \bigoplus W_3$, where W_1, W_2, W_3 are the kernels (null spaces) of $A - 2I, A + I, A^2 + I$, respectively. Note that R_2 has blocks $C(x-2), C(x+1), C(x^2+1)$, so that $R_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. $W_1 = E_2$ which is easily seen to have basis $w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $W_2 = E_{-1}$ which is easily seen to have basis $w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. W_3 is the null space of $A^2 + I$ which has basis $w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $Aw_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}$. It follows that if Q is the matrix whose columns are w_1, w_2, w_3, Aw_3, then $Q^{-1}AQ = R_2$.

Now $v = w_1 + w_2 + w_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ is a maximal vector for A acting on $\mathbb{R}^4$, so that if P is the matrix with columns v, Av, A^2v, A^3v, then $P^{-1}AP = R_1$. □

7. Use the invariant factor form of the RCF theorem to prove the elementary divisor form of the RCF theorem, as stated in the lecture of October 21. Then prove a matrix version of the elementary divisor form of the RCF theorem.

Solution. We have a finite dimensional vector space over the field F and $T \in \mathcal{L}(V, V)$. et V be a finite dimensional vector space over the field F and $T \in \mathcal{L}(V, V)$. Factor the minimal polynomial of T as $\mu_T(x) = p_1(x)^{e_1} \cdots p_r(x)^{e_r}$, where each $p_i(x) \in F[x]$ is irreducible, and set $W_i := \text{kernel}(p_i(T)^{e_i})$. Then, by the primary decomposition theorem from the lecture of Octobe 21, we have:

(i) $V = W_1 \oplus \cdots \oplus W_r$.
(ii) Each W_i is T-invariant.
(iii) $p_i(x)^{e_i}$ is the minimal polynomial of $T|_{W_i}$.

Moreover, if we, let $B_i \subseteq W_i$ be a basis for W_i, so that $B = B_1 \cup \cdots \cup B_r$ is a basis for B. If we write $A = [T]_B^B$ and $A_i = [T|_{W_i}]_{B_i}^{B_i}$, then:

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix}$$

is block diagonal. Thus, it suffices to consider $T\big|_{W_i}$, for each $1 \leq i \leq r$. So we may begin assuming $\mu_T(x) = f_1(x) = p(x)^e$, with $p(x)$ irreducible over F. By the invariant factor form of the RCF, there exists a basis $B \subseteq V$ and $f_s(x) \mid f_{s-1}(x) \mid \cdots \mid f_1(x)$ such that $[T]_B^B$ is block diagonal

$$[T]_B^B = \begin{pmatrix} C(f_1(x)) & 0 & \cdots & 0 \\ 0 & C(f_2(x)) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C(f_s(x)) \end{pmatrix}.$$

However, since each $f_i(x)$ divides $p(x)^e$, and $p(x)$ is irreducible, $f_i(x) = p(x)^{e_i}$. That $f_{i+1} \mid f_i(x)$ implies $e_i \geq e_{i+1}$. Thus, there exists $e = e_1 \geq \cdots \geq e_s \geq 1$ such that

$$[T]_B^B = \begin{pmatrix} C(p(x)^{e_1}) & 0 & \cdots & 0 \\ 0 & C(p(x)^{e_2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C(p(x)^{e_s}) \end{pmatrix},$$

which gives the elementary divisor form of the RCF for T.

For the matrix form, let $A \in \mathrm{M}_n(F)$, and define $T : F^n \to F^n$ by $T(v) = Av$, for all column vectors $v \in F^n$. Let $B = \{C_1, \ldots, C_n\} \subseteq F^n$ be a basis so that $[T]_B^B = R$ is in the elementary divisor rational canonical form. Set P to be the matrix whose columns are $C_1, \ldots, C_n$. Writing E for the standard basis of F^n we have

$$R = [T]_B^B = [I]_E^B \cdot [T]_E^E \cdot [I]_B^E = P^{-1}AP,$$

which gives what we want. □

8. Let $E = \{e_1, e_2, e_3\} \subseteq \mathbb{R}^3$ be the standard basis and suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ be such that $[T]_E^E = \begin{pmatrix} -1 & 3 & -2 \\ -1 & 3 & -4 \\ -1 & 1 & -2 \end{pmatrix}$.

(i) Find $\mu_{T,e_i}(x)$, for each e_1, e_2, e_3.
(ii) Compute $\mu_T(x)$.
(iii) Find a maximal vector for $\mathbb{R}^3$ with respect to T.

Solution. We first note that $\chi_T(x) = x^3 - 2x - 4 = (x-2)(x^2+2x+2)$.Thus, for any vector $v \in \mathbb{R}^3$, $\mu_{T,v}(x) = \chi_E(x)$, or $x^2 + 2x + 2$, or $x - 2$.

For e_1: $T(e_1) = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, T^2(e_1) = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$, so $T^2(e_1) + 2T(e_1) + 2e_1 = 0$, and there fore, $\mu_{T,e_1}(x) = x^2 + 2x + 2$.

For e_2: $T(e_2) = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, T^2(e_2) = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$. Thus, $e_2, T(e_2), T^2(e_2)$ are linearly independent. Therefore $\mu_{T,e_2}(x)$ must have degree three, therefore $\mu_{T,e_2}(x) = \chi_T(x) = x^3 - 2x - 4$. It follows that $\mu_T(x) = x^3 - 2x - 4$ and e_2 is a maximal vector for $\mathbb{R}^3$ with respect to T.

For e_3: $T(e_2) = \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}, T^2(e_3) = \begin{pmatrix} -6 \\ -2 \\ 2 \end{pmatrix}$, so that $e_3, T(e_3), T^2(_3)$ are linearly independent. Therefore, as in the previous case, $\mu_{T,e_3}(x) = x^3 - 2x - 4$ and e_3 is also a maximal vector for $\mathbb{R}^3$ with respect to T.

9. Find the singular value decomposition for the matrix $B = \begin{pmatrix} 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$.

Solution. $B = Q\Sigma P^t$, for $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and $P = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. This decomposition is obtained by first finding the eigenvalues of $B^*B = B^tB = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}$, which has characteristic polynomial $x^2(x-25)(x-2)$, and thus eigenvalues $25, 2, 0, 0$, so that the singular values of B are $\sigma_1 = 5$ and $\sigma_2 = \sqrt{2}$. This immediately gives $\Sigma = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The columns of P form an orthonormal basis of eigenvectors for B^tB, If we let v_1, v_2 be the first two columns of P, then these are eigenvectors for the non-zero eigenvalues of B^yB. The columns of Q are u_1, u_2, u_3, where $u_1 = \frac{1}{5} \cdot Bv_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{2}} \cdot Bv_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and u_3 is chosen so that u_1, u_2, u_3 are an orthonormal basis for $\mathbb{R}^3$, in which case $u_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ works.

10. Re-prove the Singular Value Theorem given in class starting with TT^*, instead of T^*T. Your solution should be almost the same as the one given in class, but there are a couple of modifications that need to be made.

Solution. We start with $T : V \to W$, for finite dimensional vector spaces V, W over F, and note that by problem 3 above, $T^* : W \to V$ exists, with the usual properties. In particular TT^* is self adjoint. Thus, TT^* is orthogonally diagonalizable and its eigenvalues are non-negative real numbers.

Step 1: Let $\lambda_1 \geq \cdots \geq \lambda_r > 0$ be the non-zero eigenvalues of TT^*.

Step 2: Let $B_W = \{v_1, \ldots, v_m\}$ be an orthonormal basis of eigenvectors for TT^*, so that $TT^*(v_i) = \lambda_i v_i$, for $1 \leq i \leq r$.

Step 3: Set $\sigma_i := \sqrt{\lambda_i}$, for $1 \leq i \leq r$.

Step 4: Set $u_1 := \frac{1}{\sigma_1}T^*(v_1), \ldots, u_r := \frac{1}{\sigma_r}T^*(v_r)$. Then $\{u_1, \ldots, u_r\}$ is an orthonormal subset of V. To see this, for $i \neq j$, we have

$$\begin{aligned}
\langle u_i, u_j\rangle_V &= \langle \frac{1}{\sigma_i}T^*(v_i), \frac{1}{\sigma_j}T^*(v_j)\rangle_V \\
&= \frac{1}{\sigma_i\sigma_j}\langle v_i, TT^*(v_j)\rangle_V \\
&= \frac{1}{\sigma_i\sigma_j}\langle v_i, \lambda_j v_j\rangle_V \\
&= \frac{\lambda_j}{\sigma_i\sigma_j}\langle v_i, v_j\rangle_V \\
&= 0,
\end{aligned}$$

and for $i = j$, we have

$$\begin{aligned}
\langle u_u, u_i\rangle_V &= \langle \frac{1}{\sigma_i}T^*(v_i), \frac{1}{\sigma_i}T^*(v_i)\rangle_V \\
&= \frac{1}{\sigma_i^2}\langle v_i, TT^*(v_i)\rangle_V \\
&= \frac{1}{\sigma_i^2}\langle v_i, \lambda_i v_i\rangle_V \\
&= \frac{\lambda_i}{\sigma_i^2}\langle v_i, v_j\rangle_V \\
&= 1.
\end{aligned}$$

Step 5: Extend $u_1, \ldots, u_r$ to $B_V = \{u_1, \ldots, u_n\}$ an orthonormal basis for V.

Step 6: By definition, $T^*(v_i) = \sigma_i u_i$, for $1 \leq i \leq r$ and $T(v_j) = 0$, for $r + 1 \leq j \leq m$. Thus, $[T^*]_{B_W}^{B_V} = \Sigma_0$, where Σ_0 is an $n \times m$ diagonal matrix whose non-zero diagonal entries are $\sigma_1, \ldots, \sigma_r$.

Step 7: By definition of T^*, from Step 6, we have $([T]_{B_V}^{B_W})^* = \Sigma_0$. Taking adjoints, and writing $\Sigma = \Sigma_0^*$, we have $[T]_{B_V}^{B_W} = \Sigma$, which is what we want. $\square$